# Finitism bottom up: a vindication of Tait

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Finitism, coming in various shades and degrees of commitment, is of both philosophical and computational interest. Hilbert proposed to view primitive-recursive arithmetic (PRA) as safely finitistic (though not necessarily exhaustive) [1, 2]. This focus on recurrence led some to contend that very broad uses of induction are also finitistic [3, 10, 11].

William Tait argued against such extensions, pointing out their impredicative nature [7, 8, 6]. However, the recurrence schema iteself has a grain of imperdicativity when refering to functions over $\mathbb{N}$ as completed totalities, and even the inductive delineation of $\mathbb{N}$ is non-finitistic as it defines $\mathbb{N}$ before admitting each of its elements. It seems desirable to construe finitism bottom-up, rather than as a top-down restriction of mathematical practice as in [4, 9].

We take as building blocks finite partial-functions over an abstract set of "atoms". Our unique basic operation is updates $f(\vec{t}) := q$ ($f$ a function, $\vec{t}, q$ closed-terms). Creation and deletion of denoted values are special cases. Every inductive data-set, such as naturals, strings, lists and trees, is representable as a cluster of finite-functions, first-order definable given finiteness.

*Elementary formulas* are generated from equations $t \simeq q$ between terms (variables allows) using connectives and quantifiers over atoms. *Concrete formulas* are of the form $\exists \vec{f}\, \varphi$ with $\varphi$ elementary. Concrete formulas are finitistically meaningful, while the exsitential quantification enables to describe processses without naming them.

Our *Concrete Theory of Finite structures (CTFS)* has, besides four trivial schemas, an Induction **Rule**: if it is *provable* that $\varphi[f]$ implies $\varphi[g]$ for any update $g$ of $f$, then $\varphi[\emptyset] \rightarrow \varphi[h]$ for any function $h$. This principle can be formulated in terms of concrete formulas only, contrary to the corresponding induction schema!

**Main Theorem:** *TCFS is mutually interpretable with PRA.* This vindicates Tait's Thesis that identifies finitism with PRA.

To prove that PRA is interpretable in CTFS we: (1) interpret equality as isomorphism; (2) show that the every PR function is interpreted by a provable function of CFTS; and (3) show that every instance of induction on equations (which by Parson's Theorem [5] is derived by the Induction Rule for existential formulas) is interpreted by the induction rule of CFTS for concrete formulas.

Conversely, we interpret CFTS in PRA by arithmetizing CTFS as a deductive system.

## References

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