MODEL-THEORETIC K_1 OF FREE MODULES OVER PIDS

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Abstract

For a first-order structure M over a language $\mathcal{L}$, Krajiček and Scanlon [KS00] defined the model-theoretic Grothendieck ring, denoted $K_0(M)$–such a ring classifies cut-and-paste equivalence classes of definable subsets (with parameters) of finite powers of M up to definable bijections. Let $\mathcal{S}(M)$ denote the groupoid whose objects are all definable subsets of M^n for $n \geq 1$, and whose morphisms are definable bijections between them. In fact, $(\mathcal{S}(M), \sqcup, \emptyset, \times, \{*\})$ is a symmetric monoidal groupoid with a pairing, where $\sqcup$ denotes disjoint union and $\times$ is the Cartesian product, and $\{*\}$ is a singleton. Given a (skeletally) small symmetric monoidal groupoid $\mathcal{S}$, Quillen gave a functorial construction of the abelian groups $(K_n(\mathcal{S}))_{n\geq 0}$, known as the K-theory of $\mathcal{S}$, which seek to classify different aspects of its objects and morphisms. Following Quillen's construction, we define the *model-theoretic K-theory* of the first order structure M by $K_n(M) := K_n(\mathcal{S}(M))$ for $n \geq 0$. In [Kub15] the model-theoretic Grothendieck rings of modules were computed and shown to be non-trivial for non-zero modules.

Bass introduced the algebraic K_1-group of a ring R, denoted $K_1^{alg}(R)$, as the direct limit of the abelianizations of the automorphism groups of finitely generated projective R-modules as their size grows larger. In other words, $K_1^{alg}(R) \cong \varinjlim_{n\in\mathbb{N}}(GL_n(R))^{ab}$. In this talk, we provide a recipe to compute the model-theoretic group $K_1(M_R)$, where M_R is a free module over a PID R, subject to the knowledge of the general linear groups $GL_n(R)$ for $n \geq 1$ [BK24], and show a surprising connection with algebraic K-theory that when R is a PID then $K_1^{alg}(R)$ naturally embeds into $K_1(R_R)$–such a connection does not hold for K_0.

We compute $K_1(M_R)$, where M_R is a free module over a large class of Euclidean domains that excludes the ring $\mathbb{Z}$ of integers, and show that $K_1(M_R) \cong K_1(R_R)$. The following table shows some concrete computations.

PID R	$K_1^{alg}(R)$	$K_1(R_R)$
Field F, char $F = 0$	$F^\times$	$\mathbb{Z}_2 \oplus \bigoplus_{i=1}^{\infty}\left((GL_i(F))^{ab} \oplus \mathbb{Z}_2\right)$
Field $F_{2^n}, n > 1$	$\mathbb{Z}_{2^n-1}$	$\mathbb{Z}_2 \oplus \bigoplus_{n=1}^{\infty}(\mathbb{Z}_{2^k-1} \oplus \mathbb{Z}_2)$
Field $F_{p^n},\ n \geq 1,\ p$ prime	$\mathbb{Z}_{p^n-1}$	$\mathbb{Z}_2 \oplus \bigoplus_{n=1}^{\infty}(\mathbb{Z}_{p^k-1} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$
Polynomial ring $F[X]$, char $F = 0$	$F^\times$	$\mathbb{Z}_2 \oplus \bigoplus_{i=1}^{\infty}\left((GL_i(F))^{ab} \oplus \mathbb{Z}_2\right)$
$\mathbb{Z}$	$\mathbb{Z}_2$	$\bigoplus_{i=0}^{\infty}\mathbb{Z}_2$

References

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