Complexity for Kleene and Action Algebras with Commutativity

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Kleene algebras are a classical object in theoretical computer science, going back to works of Kleene and Kozen [3, 4]. A Kleene algebra is a structure $(K; +, \cdot, ^*, 0, 1)$, where $(K; +, \cdot, 0, 1)$ is an idempotent semiring and a^* is the least fixpoint of $x \mapsto 1 + a \cdot x$ and $x \mapsto 1 + x \cdot a$ simultaneously. Each Kleene algebra, being a semilattice ("+" is join), enjoys a natural partial order: $a \preceq b$ iff $a + b = b$. An action algebra [12, 5], or residuated Kleene algebra, is a Kleene algebra with residuals, $\backslash$ and $/$, where $a \preceq c \mathbin{/} b \iff a \cdot b \preceq c \iff b \preceq a \mathbin{\backslash} c$. Specific subclasses of Kleene and action algebras are formed by so-called $*$-continuous ones, delimited by the following infinitary condition for Kleene star: $b \cdot a^* \cdot c = \sup\{b \cdot a^n \cdot c \mid n \geq 0\}$.

The complexity landscape for theories of Kleene algebras is depicted in [6]. Namely, while their equational theory is decidable (PSPACE-complete), both in the general and in the $*$-continuous case, already the Horn theory (reasoning from finite sets of hypotheses) is Σ^0_1-complete in the general case and Π^1_1-complete in the $*$-continuous case. For hypotheses without Kleene star, the latter is lowered to Π^0_2-completeness. For action algebras, already the equational theory is undecidable, being Π^0_1-complete in the $*$-continuous case [2, 11] and Σ^0_1-complete in the general one [7].

In this talk, we present three recent results by the author, concerning complexity for Kleene and action algebras which are commutative (i.e., $a \cdot b = b \cdot a$ for any $a, b \in K$) or partially commutative.

- The Horn theory of commutative $*$-continuous Kleene algebras is Π^1_1-complete. The fragment of this Horn theory with $*$-free hypotheses is Π^0_2-complete. [10]
- The equational theory of all commutative action algebras and that of all $*$-continuous commutative action algebras are, respectively, Σ^0_1- and Π^0_1-complete. [8]
- Reasoning from commutativity conditions (i.e., a finite set of hypotheses of the form $x \cdot y = y \cdot x$) on the class of all Kleene algebras is Σ^0_1-complete. [9] Independently, undecidability was also proved by Azevedo de Amorim et al. [1]. For the $*$-continuous case, Π^0_1-completeness of this problem was previously known [6].

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