Ordered Fragments of First-Order Logic or: How To Make Your Favourite Logic Decidable?

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FLUTED LOGIC

The *fluted fragment* (denoted $\mathcal{FL}$) is a fragment of first-order logic (denoted $\mathcal{FO}$) in which, roughly put, variables appear in predicates following the order in which they were quantified. For illustrative purposes, we translate the sentence "Every conductor nominates their favorite soloist to play at every concert" into the language described above as follows: $\forall x_1[cond(x_1) \rightarrow \exists x_2\langle solo(x_2) \wedge fav(x_1, x_2) \wedge \forall x_3(conc(x_3) \rightarrow nom(x_1, x_2, x_3))\rangle]$. More formally, the key rule of flutedness is as follows. If $x_1, \ldots, x_\ell$ are quantified (in order) leading up to some atom α, then $\alpha \equiv p(x_i, \ldots, x_\ell)$ for some $1 \leq i \leq \ell$; that is to say, α can only feature an ordered suffix of $x_1, \ldots, x_\ell$ as arguments. Because of this restriction sentences axiomatising transitivity, symmetry and reflexivity are not in $\mathcal{FL}$.

The fluted fragment [15] is a member of *ordered logics* – a family of decidable (in terms of satisfiability) fragments of first-order logic which also includes the *ordered* [9, 10], *forward* [2] and (the most recent addition) *adjacent* [4] fragments. In the sequel we will disregard the ordered and forward fragments seeing that the fluted fragment, as it turns out, is at least as expressive as both of them [3].

The fluted fragment is of particular interest as it is robustly decidable in terms of satisfiability even in the presence of extensions such as *counting quantifiers* $\exists_{[\geq m]}x.\varphi$ which state "the number of elements satisfying φ is m or greater" [14] and *periodic counting quantifiers* $\exists_{[m^{+p}]}x.\varphi$ stating "the number of elements satisfying φ is m, or $m+p$, or $m+2p$, or ..." [11]. (Do note that the latter is not $\mathcal{FO}$-expressable). There is a limit, however, of how far one can go in terms of extensions. A notable undecidable augmentation is that of a *Härtig quantifier* $I(x, y)(\varphi, \psi)$ – a (non-$\mathcal{FO}$) extension which allows comparison of cardinalities of sets defined by φ and ψ.

No matter which decidable extension one picks, the high-level proof idea (for decidability of satisfiability) is the same: given a sentence in the language exploit the ordered nature of flutedness and reduce the number of variables until an "easy enough" base case is reached. This variable reduction procedure will be the key focus throughout the talk. We will also establish a new result from [11]: when one is concerned with fluted languages, only models in which elements behave (in a sense that will be made clear in the talk) *homogenously* need be considered. This, as it will become apparent, will greatly reduce the complexity of decidability arguments.

THE ADJACENT FRAGMENT

Suppose α is an atom in a context where variables $x_1, \ldots, x_\ell$ are quantified in order. We say that α is adjacent if it takes the form $p(x_{i_1}, \ldots, x_{i_k})$, where $|i_j - i_{j+1}| \leq 1$ for $1 \leq j < k$ and $1 \leq i_j \leq \ell$. That is to say, subsequent arguments in α are neighboring elements of the sequence $x_1, \ldots, x_\ell$. We call the language with the adjacency condition imposed the *adjacent fragment* (denoted $\mathcal{AF}$). Clearly, $\mathcal{FL} \subseteq \mathcal{AF}$. As opposed to the fluted fragment, symmetry and reflexivity (amongst other properties) is axiomatisable in the new language, thus making the adjacent fragment more expressive. Note that the adjacent fragment also subsumes (up to logical equivalence) the two-variable fragment of first-order logic $\mathcal{FO}^2$.

What is gained in expressive power by generalising the flutedness condition, however, is lost in terms of decidability. Whilst in the presence of no extensions the satisfiability problem is decidable using a variable reduction technique similar to that as for the fluted fragment [4], the homogenous model property is lost. Undecidability of satisfiability follows from [11] for the adjacent fragment with (periodic) counting quantifiers. It is currently open whether the satisfiability problem for $\mathcal{AF}^3$ with counting quantifiers is decidable. The finite variant of the satisfiability problem for $\mathcal{AF}^3$ with counting quantifiers, however, is Σ^0_1-complete [11]. In the talk we will discuss what makes the problem 'tricky' and undecidable when an additional variable is permitted. By doing so, we will give intuition as to why the variable reduction procedure fails.

Table 1 provides the currently known upper and/or lower bounds for the satisfiability problems of ordered languages discussed.

	$\mathcal{FO}^1$	$\mathcal{FO}^2$	$\mathcal{AF}^3$	$\mathcal{AF}^\ell$	$\mathcal{FL}^3$	$\mathcal{FL}^\ell$
no extensions	NP-c [folklore]	NExp-c [8]	NExp-c [4]	$(\ell-2)$-NExp [4]	NExp-c [15]	$(\ell-2)$-NExp [15]
"=" predicate	NP-c [folklore]	NExp-c [8]	2-NExp [5]	$(\ell-1)$-NExp [5]	2-NExp [14]	$(\ell-1)$-NExp [14]
$\exists_{[\geq m]}x.\varphi$	NP-c [12]	NExp-c [13]	???	Π^0_1-c [11]	2-NExp [14]	$(\ell-1)$-NExp [14]
$\exists_{[m^{+p}]}x.\varphi$	NP-c [1]	2-NExp [6]	Σ^0_1-h [11]	Σ^1_1-c [11]	2-NExp [11]	$(\ell-1)$-NExp [11]
$I(x,y)(\varphi,\psi)$	NP-c [1]	Σ^1_1-h [7]	Σ^1_1-h [7]	Σ^1_1-h [7]	Σ^1_1-h [7]	Σ^1_1-h [7]

Table 1: Complexity and (un)decidability of the satisfiability problem for ordered languages (in the top row) under various extensions (on the left-most column). All complexity classes are in regard to time. C-c (resp. -h) stands for complete (resp. hard). In all cases $\ell \geq 4$.

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