The Good Company Problem, pluralism, and the foundations of mathematics

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In this paper, I plan to assess the abstraction principles that form the "good company" of the Cantor-Hume Principle (see Mancosu (2016)) according to their consequences for the foundations of mathematics. The main goal of Frege's logicist program was to reduce arithmetic to logic, thus founding mathematics on logic alone (Frege (1893-1903)). However, his use of Basic Law V and of the Cantor-Hume Principle (CHP) led to Russell's Paradox. After Wright (1983) showed that there were ways to salvage Frege's program, there was a resurgence of interest on abstraction principles, culminating in the proof that in some cases abstraction principles can be used to derive the axioms of PA_2 (second order arithmetic) from impredicative second-order logic without falling into Russell's Paradox (Heck (2011)). Mancosu (2016) argued that there are several of these "good" abstraction principles, and that this poses the "Good Company Problem", that is, the problem of choosing which, among all these abstraction principles, is the "best" or "right" one. In the same book, Mancosu argued that there are 3 possible answers to the Good Company Problem: a *conservative* neo-logicist will argue that only the CHP is acceptable, a *moderate* neo-logicist might turn to the finite version of CHP (F-CHP) as more suitable, and finally a *liberal* neo-logicist might argue that all good abstraction principle are mostly equivalent, since they can all be used to reduce arithmetic to logic.

More recently, Sereni et al. (2023) argued that another possible reply to the Good Company Problem is a *pluralist* one. According to this argument, there are ways to accept *all* the abstraction principles that for the "good company" of the CHP: conceptual pluralism, domain pluralism, and pluralism about criteria. For the *conceptual pluralist*, each abstraction principle instantiates a different, and equally legitimate, conception of cardinal number and size, just like different logics might give us different conceptions of validity (see, for example, Beall and Restall (2006)). According to *domain pluralism*, each different abstraction principle gives rise to different domains for arithmetic, that are all equally consistent and thus can co-exist. Finally, *pluralism about criteria* states that there are different criteria that we use to choose which abstraction principle to adopt, but since they are all equally good in different contexts, there is no way to choose between these criteria, and thus no way to differentiate between abstraction principles. All in all, the pluralist answer to the Good Company Problem is clear: all good abstraction principles can be adopted at the same time.[1]

In this talk, I plan to assess the different abstraction principles that form the "good company" of CHP (so its finite version FCHP, the Peano Principle, and the Boolos' Principles) against the logicist goal of developing a foundation of mathematics. I will show in particular that while all these abstraction principles are equivalent when considered as a foundation of arithmetic, the set theory they give rise to are all quite different. In particular, I show that only the Cantor-Hume Principle can be actually used as the stepping stone for a foundation of mathematics, since it is the only one compatible with full ZF. The other ones all give rise to weaker fragments of set theory (General Set Theory, from Boolos and Thomson (1987), and "Euclidean" set theories, see Parker (2013)), that cannot be used as a foundation of (classical) analysis or geometry (assuming the most minimal set of axioms). I claim that this fact makes the liberal and moderate neo-logicist positions less plausible, while making the pluralist position a good possible answer to the Good Company Problem iff it is coupled with proper set-theoretic pluralism.

[1] Sereni et al. (2023) explains the difference between pluralism and liberal neo-logicism as follows: while for the liberal neo-logicist the choice doesn't matter, for the pluralist there is no need to make the choice in the first place.

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