The proof theory of pro-aperiodic terms

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An aperiodic monoid is a monoid M such that for every element x, there exists a positive integer n such that $x^n = x^{n+1}$. Aperiodic monoids play a fundamental role in finite semigroup theory and automata theory. For example, Schützenberger's seminal theorem [5] states that the aperiodic monoids recognise precisely the class of star-free languages.

1 Pro-aperiodic terms

Definition 1 (Pro-aperiodic metric)**.** Let A be a finite alphabet. An aperiodic monoid M separates two words u and v of A^* if there exists a morphism $\varphi : A^* \to M$ such that $\varphi(u) \neq \varphi(v)$. Define

$$d(u, v) := 2^{-\min\{|M| \mid M \text{is an aperiodic monoid that separates } u \text{ and } v\}}.$$

From the observation that any two words can be separated by a finite aperiodic monoid, it is easy to obtain that pro-aperiodic metric is indeed a metric (in fact, an ultrametric). The set of pro-aperiodic words $\widehat{A^*}$ is the completion of A^* with the pro-aperiodic metric.

It is relatively difficult to give "concrete" examples of pro-aperiodic words which are not words. Two such examples are x^ω and $x^{\omega+1}$, associated with every pro-aperiodic word x, defined as follows: $x^\omega := \lim_{n\to\infty} x^{n!}$ $\quad x^{\omega+1} := \lim_{n\to\infty} x^{n!+1}$. Immediately, we have $x^{\omega+1} = x.x^\omega = x^\omega.x$. Moreover, we have $x^\omega = x^{\omega+1}$.

Definition 2 (Pro-aperiodic terms)**.** The set of pro-aperiodic terms are given by, $s, t ::= a \in \Sigma \mid s \cdot t \mid t^\omega$.

Note that several pro-aperiodic words are not described by any pro-aperiodic term (just by a cardinality argument). However, it is possible to axiomatise the equational theory of pro-aperiodic terms and consequently, decide the word problem.

2 Axiomatisation of pro-aperiodic terms

McCammond [4] gave a sound and complete axiomatisation of the equational theory of pro-aperiodic terms:

$$(st)u = s(tu) \qquad (t^\omega)^\omega = t^\omega \qquad \forall n \in \mathbb{N}.(t^n)^\omega = t^\omega \qquad t^\omega t^\omega = t^\omega \qquad t^\omega t = tt^\omega = t^\omega \qquad (st)^\omega s = s(ts)^\omega$$

Based on this axiomatisation, McCammond showed that every pro-aperiodic term has a unique normal form that can be effectively computed. This implies that the word problem is decidable. Since then, several works [3, 1, 2] have proved the decidability of the word problem using various techniques and improved the complexity of the problem. We are interested in McCammond's axiomatisation and want to study it using modern proof-theoretic tools. We first show that pro-aperiodic **cannot be finitely axiomatised**.

Our main result is that we propose a sound and complete **non-wellfounded proof system for pro-aperiodic terms**. There are two peculiarities of this system: (a) two entailment relations corresponding to prefix and suffix respectively (b) the progress condition asserts that for all branches *every* thread is progressing (as opposed to a ∀∃ condition). We give an example of an inference rule that mixes prefix and suffix.

$$\frac{\Gamma \to_{pref} \Delta \qquad \Delta \to_{suff} \Gamma \qquad \Gamma' \to_{pref} \Delta'}{\Gamma, \Gamma' \to_{pref} \Delta, \Delta'}$$

We end with a few open questions: (a) Can we have a complete cyclic system (i.e. restricting non-wellfounded trees to regular trees)? (b) Can we obtain MacCammond's axiomatisation from our proof system? (c) Is there quasi-equational (i.e. allowing conditional equations) axiomatisation of the theory of pro-aperiodic terms?

References

[1] Samuel J. van Gool and Benjamin Steinberg. "Pro-Aperiodic Monoids via Saturated Models". In: *34th Symposium on Theoretical Aspects of Computer Science (STACS 2017)*. Ed. by Heribert Vollmer and Brigitte Vallée. Vol. 66. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2017, 39:1–39:14. ISBN: 978-3-95977-028-6. DOI: 10.4230/LIPIcs.STACS.2017.39. URL: https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.STACS.2017.39.

[2] Samuel J. van Gool and Benjamin Steinberg. "Pro-aperiodic monoids via saturated models". In: *Israel Journal of Mathematics* 234.1 (Oct. 2019), pp. 451–498. DOI: 10.1007/s11856-019-1947-6. URL: https://doi.org/10.1007/s11856-019-1947-6.

[3] Martin Huschenbett and Manfred Kufleitner. "Ehrenfeucht-Fraïssé Games on Omega-Terms". In: *31st International Symposium on Theoretical Aspects of Computer Science (STACS 2014)*. Ed. by Ernst W. Mayr and Natacha Portier. Vol. 25. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2014, pp. 374–385. ISBN: 978-3-939897-65-1. DOI: 10.4230/LIPIcs.STACS.2014.374. URL: https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.STACS.2014.374.

[4] J. P. McCammond. "Normal forms for free aperiodic semigroups". In: *International Journal of Algebra and Computation* 11.05 (2001), pp. 581–625. DOI: 10.1142/S0218196701000693. eprint: https://doi.org/10.1142/S0218196701000693. URL: https://doi.org/10.1142/S0218196701000693.

[5] M.P. Schützenberger. "On finite monoids having only trivial subgroups". In: *Information and Control* 8.2 (1965), pp. 190–194. ISSN: 0019-9958. DOI: https://doi.org/10.1016/S0019-9958(65)90108-7. URL: https://www.sciencedirect.com/science/article/pii/S0019995865901087.