On Inferential Many-valuedness, Many-valued Logical Structures and Suszko's Thesis

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A *logical structure* (see [1]) is a pair $(\mathscr{L}, \vdash)$, where $\mathscr{L}$ is a set, $\mathcal{P}(\mathscr{L})$ denotes the power set of $\mathscr{L}$ and $\vdash \subseteq \mathcal{P}(\mathscr{L}) \times \mathscr{L}$. A logical structure $(\mathscr{L}, W)$ is said to be a *Tarski-type logical structure* if W satisfies the following well-known conditions of *reflexivity, monotonicity* and *transitivity*. It is well known that a logical structure is of Tarski-type iff it has an adequate bivalent semantics (see [9, Theorem 2.4] for details).

Suszko's Thesis (henceforth **ST**) is the (philosophical) claim that "there are but two logical values, true and false" (see [3]). This thesis has been given formal contents by the so-called *Suszko Reduction* (see [10]). However, as has been pointed out in [2], "there seems to be no paper where Suszko explicitly formulates (SR)[the Suszko Reduction] in full generality!" Furthermore, there seems to be no *explicit definition* of 'logical value'. This makes **ST** rather ambiguous, difficult to justify and open to interpretation.

However, interpretations differ. For example, the 'real' reason behind the existence of adequate bivalent semantics for every Tarski-type logical structure, according to Malinowski, corresponds to the number of divisions of the universe of interpretation (see [6]). This particular conceptualisation of 'logical value' is termed by Malinowski as *inferential many-valuedness*. This notion of many-valuedness is different from the usual conception of many-valuedness − *algebraic many-valuedness*. Even though much research has been done on inferential many-valuedness, surprisingly there is no *explicit* or *formal* definition of the same! The underlying intuition has been provided several times (see [5, 7, 8] e.g.), of course. The lack of a formal definition, however, is problematic if, e.g., one wants to characterise the class of inferentially n-valued structures for a natural number n.

One aim of this talk is to remedy this situation. Based on a rather general notion of *semantics*, we define inferential many-valuedness and *prove* that the class of monotonic logical structures can be segregated into four sub-classes (not necessarily disjoint), each of which has a certain type of adequate 3-valued semantics. Moreover, we identify several classes of logical structures, for which these 3-valued semantics are respectively minimal. It turns out that Malinowski's notion of many-valuedness is just a particular one among a galaxy of such possible notions.

Next, we introduce another notion of semantics and show that several logical structures can have adequate bivalent semantics in a very specific sense. We illustrate this first by providing adequate bivalent semantics for monotonic logical structures and then for certain particular types of non-monotonic logical structures. We also discuss some problems regarding the existence of adequate bivalent semantics for *arbitrary* non-monotonic logical structures.

Finally, we delve into a deeper analysis of the *principle of bivalence* and explicate how **ST** is related to it. The ensuing analysis culminates in the proposal that many-valued logical structures can be obtained if one considers the language/metalanguage hierarchy also. This helps to build formal bridges between the theory of graded consequence (see [4]) and the theory of many-valued logical structures, leading to generalisations of **ST**.

References

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