Real analysis via logic

Steve Vickers
School of Computer Science, University of Birmingham
s.j.vickers.1@bham.ac.uk

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Abstract

What is a "collection"? All too readily one assumes that it ought to form a set, or at least a class, but of course this presupposes a given set theory (classical or otherwise). If instead we describe the elements of a collection as the *models of a logical theory*, then a single description can serve to describe the collection in a range of set theories.

A hugely important example is that of topological spaces. For some time now, constructive mathematics has tried to teach us that these are best approached not point-*set*, with a given *set* of points, but "point-free", with a different structure such as a locale [Joh82] or a formal topology [Sam87]. Then we have (Joyal and Tierney [JT84]) that spaces in the internal maths of a topos of sheaves are equivalent to bundles over the corresponding space. This is an excellent result that has no good counterpart in point-set topology, and it leads to a principle that we can rigorously understand bundles as continuously indexed families of spaces, provided we reason point-free and constructively.

Using those structures directly is unintuitive and lacks transparency. However, they turn out to be ways to present logical theories, and techniques have gradually become available for reasoning with them transparently as collections of models – this has been the motivation for much of my own work (eg [Vic99, Vic07, Vic22]). The *geometric* logic used (see [Joh02, D1.1]) is constructive. Moreover, its connectives are restricted to conjunctions and (possibly infinitary) disjunctions, which correspond to intersections and unions used in axiomatizing open sets of topology.

I shall give a taster of the principles and techniques by which this approach has recently started to be applied to real analysis. In Ming Ng's thesis he defined real exponentiation and logarithms (see [NV22]), and subsequently [Vic23] I have given an account of the Fundamental Theorem of Calculus, exploiting earlier results on integration [Vic08] and differentiation [Vic09].

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