# THE HARTOGS–LINDENBAUM SPECTRUM OF SYMMETRIC EXTENSIONS (ABSTRACT)

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Let $X$ be a set. The *Hartogs number* of $X$ is

$$\aleph(X) \coloneqq \min\{\alpha \in \mathrm{Ord} \mid \text{ There is no injection } f\colon \alpha \to X\}.$$

The *Lindenbaum number* of $X$ is

$$\aleph^*(X) \coloneqq \min\{\alpha \in \mathrm{Ord} \setminus \{0\} \mid \text{ There is no surjection } f\colon X \to \alpha\}.$$

Under the axiom of choice, any set can be well-ordered and so we obtain $\aleph(X) = \aleph^*(X) = |X|^+$ for all $X$. However, this uses $\mathsf{AC}$ in an essential way. Indeed, the only relation proved by $\mathsf{ZF}$ is that $\aleph(X) \leqslant \aleph^*(X)$ are both cardinals, and $\aleph(X)$ is finite if and only if $\aleph(X) = \aleph^*(X) = |X|^+$: In [1] the authors show that for all infinite $\lambda \leqslant \kappa$ there is a symmetric extension in which for some set $X$, $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$. Indeed, this can also be done *simultaneously*: $\mathsf{ZF}$ is consistent with "for all infinite $\lambda \leqslant \kappa$ there is $X$ such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$". This leads one to the following question:

**Question.** What are the possible Hartogs–Lindenbaum spectrums of models of $\mathsf{ZF}$?

We shall partially classify such spectra using technology that pushes eccentric sets (those $X$ with $\aleph(X) < \aleph^*(X)$) upwards to produce eccentric $Y$ with large $\aleph(Y)$. As a consequence, in models of $\mathsf{SVC}$, the Hartogs and Lindenbaum numbers of sets eventually 'stabilise' in cofinality:

**Theorem.** *There is a set $C$ of cardinals and a cardinal $\Omega$ such that for all $\kappa \geqslant \lambda > \Omega$, there is $X$ such that $\aleph(X) = \lambda$ and $\aleph^*(X) = \kappa$ if* and only if*:*

- *$\lambda = \kappa$ is a successor; or*
- *$\mathrm{cf}(\lambda) \in C$ and $\kappa = \lambda^+$.*

## References

1. Asaf Karagila and Calliope Ryan-Smith, *Which pairs of cardinals can be Hartogs and Lindenbaum numbers of a set?*, arXiv **2309.11409** (2023).

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